Number Theory 8

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Q) For how many values of k is 12^{12} the LCM of 6^6, 8^8 and k, $k \in \mathbb{Z}$

Ans:— $12^{12} = 2^{24} \times 3^{12}$

$k = 2^a \times 3^b$, $6^6 = 2^6 \times 3^6$, $8^8 = 2^{24}$

$\operatorname{lcm}(2^a \times 3^b, 2^6 \times 3^6, 2^{24}) = 2^{24} \times 3^{12}$

$b = 12$, $a = \{0, 1, \ldots, 24\}$ $\Rightarrow$ k has 25 elements

Q) Find all positive integers n such that

$$(3^{n-1} + 5^{n-1}) \mid (3^n + 5^n)$$

Ans:— $3^{n-1} + 5^{n-1} \mid 5(3^{n-1} + 5^{n-1}) = 3^n + 2 \times 3^{n-1} + 5^n$

$\Rightarrow 3^{n-1} + 5^{n-1} \mid (3^n + 2 \cdot 3^{n-1} + 5^n) - (3^n + 5^n)$

$\Rightarrow 3^{n-1} + 5^{n-1} \mid 2 \cdot 3^{n-1}$

But for $n > 1$, $5^{n-1} > 3^{n-1} \Rightarrow 3^{n-1} + 5^{n-1} > 2 \cdot 3^{n-1}$

$\Rightarrow 3^{n-1} + 5^{n-1} \nmid 2 \cdot 3^{n-1}$ $\qquad a \mid b \Rightarrow |a| \le |b|$

$\Rightarrow 3^{n-1} + 5^{n-1} \nmid 3^n + 5^n$ for $n > 1$

For $n = 1$,

$2 \mid 8$

Q) Prove that for positive integer n, $\exists$ a positive integer m such that each term of $m+1$, $m^m + 1$, $m^{m^m} + 1$, $\ldots$ is divisible by n. $n > 1$

of $m+1, m^m+1, m$ ...

Ans:— $n \mid m+1$, $n \mid m^m+1$, $n \mid m^{m^m}+1$.

$kn = m+1$, $n \mid (kn-1)+1$, $n \mid (kn-1)^{(kn-1)}+1$, ...
$m = (kn-1)$,

$(kn-1)^{kn-1} \rightarrow$ last term will be -1 when $kn-1$ is odd $\rightarrow$ when $k=2$

$m = 2n-1$ will suffice

$(2n-1)^{2n-1} = nk-1$, $(2n-1)^{(2n-1)^{(2n-1)}} = (2n-1)^{nk-1}$
is odd as $(2n-1)$ is odd $= \underbrace{nk'-1}_{\text{again odd}}$

So $n \mid m^{m^{m\cdots}}+1$ for $m = 2n-1$

Q) Show that the cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an arithmetic progression

Ans:— $a, a+d, a+2d, \ldots$
(HomeWork) Hints:— P_1, P_2, P_3 , $\sqrt[3]{P_3} - \sqrt[3]{P_2} = k_1 d$
$\sqrt[3]{P_2} - \sqrt[3]{P_1} = k_2 d$

Q) Prove that the sum and product of two relatively prime integers are themselves relatively prime

Ans:— $\gcd(a, b) = 1$
$\gcd(ab, b) = b$
$\gcd(ab, a) = a$
Suppose a prime $p \mid ab \Rightarrow p \mid a$ or $p \mid b$ but not together
[illegible] $\Rightarrow p \nmid (a+b)$

Suppose a prime $p \mid ab \Rightarrow p \mid a$ or $p \mid b$

if $p \mid a$ and $p \nmid b \Rightarrow p \nmid (a+b)$

if $p \nmid a$ and $p \mid b \Rightarrow p \nmid (a+b)$

$\Rightarrow p \mid ab \Rightarrow p \nmid (a+b)$